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$\mathsf{C}_p(X)$ and Arhangel'skiĭ's α_i-spaces

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Abstract

Nogura showed that whereas Arhangel'skiĭ's properties α_1, α_2 and α_3 are preserved by finite products, the property α_4 is not. It is shown here that for each space X the properties α_2, α_3 and α_4 are the same for the function space $\mathsf{C}_p(X)$. As a consequence, α_4 is closed under finite products of such function spaces. © 1998 Elsevier Science B.V. All rights reserved.

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Let X be an infinite completely regular Hausdorff space. The Cartesian product of X copies of the real line $\mathbb{R}$, which is the set of all functions from X to $\mathbb{R}$, is endowed with the Tychonoff product topology and is denoted by $\mathbb{R}^X$. The set of *continuous* functions from X to $\mathbb{R}$ endowed with the topology which it inherits as subset of $\mathbb{R}^X$, is denoted by $\mathsf{C}_p(X)$; the topology is said to be the *topology of pointwise convergence*.

For x an element of X define the following notation:

Γ_x: the set of $A \subset X \setminus \{x\}$ such that A is countably infinite, and each neighborhood of x contains all but finitely many elements of A. These can be viewed as the nontrivial sequences which converge to x;

Ω_x: the set of $A \subset X \setminus \{x\}$ such that x is in the closure of A.

A space has *countable tightness* if for any x, each element of Ω_x has a countable subset which is a member of Ω_x. A space has the *Fréchet property* if for each x, each element of Ω_x has a subset which is an element of Γ_x. The Fréchet property is often also called the Fréchet–Urysohn property. A space is *sequential* if for each subset Y which is not closed, there there is an x not in Y such that Y contains a member of Γ_x.

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In [1] Arhangel'skiĭ introduced the following properties:

α_1: a space has property α_1 if for each each x and for each sequence $(O_n\colon n \in \mathbb{N})$ of elements Γ_x, there is a single element O of Γ_x, such that for each n the set $O_n \setminus O$ is finite;

α_2: a space has property α_2 if there is for each x, for each sequence $(O_n\colon n \in \mathbb{N})$ from Γ_x, a B in Γ_x, such that for each n, $B \cap O_n$ is infinite;

α_3: a space has property α_3 if there is for each x and each sequence $(O_n\colon n \in \mathbb{N})$ from Γ_x, an element A of Γ_x such that for infinitely many n, $A \cap O_n$ is infinite;

α_4: a space has property α_4 if there is for each x and each sequence $(O_n\colon n \in \mathbb{N})$ from Γ_x, a B in Γ_x such that for infinitely many n, $B \cap O_n$ is nonempty.

Each of these α_i-properties implies the next one.

The following two selection hypotheses are convenient expository devices: Let S be an infinite set and let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of S. Then the symbol $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the following selection hypothesis: For every sequence $(O_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(T_n\colon n \in \mathbb{N})$ such that for each n, $T_n \in O_n$, and $\{T_n\colon n \in \mathbb{N}\} \in \mathcal{B}$. A second, related selection hypothesis is denoted $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$, and differs from $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ in that for each n the T_n is required to be a finite subset of O_n, and $\bigcup_{n=1}^{\infty} T_n$ is required to be an element of $\mathcal{B}$.

The purpose of this note is to discuss to what extent the α_i-properties are distinguished from each other by spaces of the form $\mathsf{C}_p(X)$ (Sections 1 and 2), and to describe their relation to the tightness properties and the Fréchet properties (Section 3). In the course of the discussion three cardinal numbers, $\mathfrak{p}$, $\mathfrak{t}$ and $\mathfrak{b}$, make their appearance. Van Douwen's article [23] is an excellent reference regarding these.

1. $\mathsf{C}_p(X)$ and the α_2, α_3 and α_4-properties

Let $\mathbf{0}$ denote the function on X which is equal to zero everywhere. Since $\mathsf{C}_p(X)$ is homogeneous, it has property $\mathsf{S}_1(\Gamma_x, \Gamma_x)$ if, and only if, it has property $\mathsf{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$. For f an element of $\mathsf{C}_p(X)$, the symbol $|f|$ denotes the function which computes the absolute value of values of f. $\mathsf{C}_p(X)$ is mapped to $\mathsf{C}_p(X)$ by the absolute value operation.

Lemma 1. *Let X be a space.*

(1) *For each sequence $(f_n\colon n \in \mathbb{N})$ of elements of $\mathsf{C}_p(X)$ the following are equivalent:*
   (a) *$(f_n\colon n \in \mathbb{N})$ is in $\Gamma_{\mathbf{0}}$;*
   (b) *$(|f_n|\colon n \in \mathbb{N})$ is in $\Gamma_{\mathbf{0}}$.*

(2) *If for each $i \leqslant m$ we have $(f_n^i\colon n \in \mathbb{N})$ and for each $n \in \mathbb{N}$ and $x \in X$ we have $f_n^i(x) \geqslant 0$, then the following are equivalent:*
   (a) *For each $i \leqslant m$, $(f_n^i\colon n \in \mathbb{N})$ is in $\Gamma_{\mathbf{0}}$;*
   (b) *$(\sum_{i \leqslant m} f_n^i\colon n \in \mathbb{N})$ is in $\Gamma_{\mathbf{0}}$.*

(3) *If $\mathsf{C}_p(X)$ has property $\mathsf{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$, then it has property α_2.*

Proof. The proofs for (1) and (2) are left to the reader. For (3): Note that if for each n the sequence $(f_k^n\colon k \in \mathbb{N})$ is an element of Γ_0, then we can make up new sequences $((g_k^n\colon k \in \mathbb{N})\colon n \in \mathbb{N})$ such that for each n there are infinitely many m with $(g_k^m\colon k \in \mathbb{N}) = (f_k^n\colon m < k,\ k \in \mathbb{N})$. If we now apply $\mathsf{S}_1(\Gamma_0, \Gamma_0)$ to the sequences $((g_k^m\colon k \in \mathbb{N})\colon m \in \mathbb{N})$ we find a sequence $(g_{k_m}^m\colon m \in \mathbb{N})$ in Γ_0, and it contains infinitely many terms from each $(f_k^n\colon k \in \mathbb{N})$, as required by α_2. □

Theorem 2. *Let X be a space. Then $\mathsf{C}_p(X)$ has property α_2 if, and only if, it has property α_4.*

Proof. We must show that if $\mathsf{C}_p(X)$ has property α_4, then it has property α_2. For each n let a sequence $(f_k^n\colon k \in \mathbb{N})$ from Γ_0 be given. By part (1) of Lemma 1 we may assume that for each n and k, and for each $x \in X$, $f_k^n(x) \geqslant 0$.

For each m and k define:

$$g_k^m := \sum_{i=1}^{m} f_k^i.$$

By part (2) of Lemma 1 for each m the sequence $(g_k^m\colon k \in \mathbb{N})$ is an element of Γ_0. Apply the property α_4 to the family $((g_n^m\colon n \in \mathbb{N})\colon m \in \mathbb{N})$. We find sequences $m_1 < m_2 < m_3 < \cdots$ and $n_1, n_2, n_3, \ldots$ of natural numbers such that $(g_{n_k}^{m_k}\colon k \in \mathbb{N})$ is an element of Γ_0.

Put $m_0 = 0$, and then define for each i a k_i by:

$$f_{k_i}^i := f_{n_j}^i \quad \text{where } m_{j-1} < i \leqslant m_j.$$

For each i let $\phi(i)$ be the unique j with $m_{j-1} < i \leqslant m_j$. Then $\phi(i) \leqslant \phi(i')$ whenever $i < i'$, and ϕ is finite to one. Notice that for each x, and for each i we have:

$$0 \leqslant f_{k_i}^i(x) \leqslant g_{n_{\phi(i)}}^{m_{\phi(i)}}(x).$$

Since $(g_{n_{\phi(i)}}^{m_{\phi(i)}}(x)\colon i \in \mathbb{N})$ converges to zero, so does $(f_{k_i}^i(x)\colon i \in \mathbb{N})$. It follows that $(f_{k_i}^i\colon i \in \mathbb{N})$ is a member of Γ_0. We have shown that $\mathsf{C}_p(X)$ satisfies property $\mathsf{S}_1(\Gamma_0, \Gamma_0)$. Now apply part (3) of Lemma 1. □

In [13] Nogura proved theorems which imply that if Hausdorff spaces X and Y are both α_i for an i in $\{1,2,3\}$, then so is $X \times Y$. In [14] he gave an example of compact Fréchet spaces X and Y such that $X \times Y$ is not an α_4-space. A result of Olson [15] together with a result of Arhangel'skiĭ [1] imply that compact Fréchet spaces are α_4. Since Nogura also showed in [14] that the product of an α_3-space with an α_4-space is an α_4-space, his example gives spaces which are compact Fréchet, so α_4, but not α_3 and also shows that the product of two compact α_4-spaces need not be α_4.

Theorem 2 shows that none of these phenomena can be witnessed by spaces of the form $\mathsf{C}_p(X)$. In particular:

Corollary 3. *Let X and Y be spaces. If $\mathsf{C}_p(X)$ and $\mathsf{C}_p(Y)$ are α_4-spaces, so is $\mathsf{C}_p(X) \times \mathsf{C}_p(Y)$.*

Proof. If $\mathsf{C}_p(X)$ and $\mathsf{C}_p(Y)$ are α_4-spaces, then they are α_2-spaces. By Nogura's theorem, $\mathsf{C}_p(X) \times \mathsf{C}_p(Y)$ is an α_2-space. But an α_2-space is an α_4-space. □

2. $\mathsf{C}_p(X)$ and the property α_1

To gain some insight into the α_1-property in the context of $\mathsf{C}_p(X)$, we recall another concept from the literature: A sequence $(f_n\colon n \in \mathbb{N})$ of real-valued functions on a space X converges *quasi-normally* to f if there exists a sequence $(\varepsilon_n\colon n \in \mathbb{N})$ of positive real numbers such that

- $\lim_{n\to\infty} \varepsilon_n = 0$, and
- for each x, for all but finitely many n, $|f_n(x)| < \varepsilon_n$.

The term "quasi-normal convergence" was introduced by Bukovská and studied by her in [5]. Earlier, quasi-normal convergence was called "equal convergence" by Császár and Laczkovich [7]. In [6] a space X is said to be a QN-space if whenever a sequence $(f_n\colon n \in \mathbb{N})$ of continuous real-valued functions on X converges pointwise to the continuous function f, then the convergence is in fact quasi-normal convergence.

Theorem 4. *If $\mathsf{C}_p(X)$ is an α_1-space, then X is a QN-space.*

Proof. Let $(f_n\colon n \in \mathbb{N})$ be a sequence in $\mathsf{C}_p(X)$ which converges pointwise to the zero function. For each k and n, define

$$f_n^k(x) = \sqrt[k]{|f_n(x)|} + \frac{1}{k \cdot n}.$$

Then for each k, $(f_n^k\colon n \in \mathbb{N})$ is a sequence in $\mathsf{C}_p(X)$ which converges pointwise to the zero function.

Apply α_1 and choose $(g_n\colon n \in \mathbb{N})$ in $\mathsf{C}_p(X)$ and for each k an n_k such that

(1) $n_1 < \cdots < n_k < \cdots$;

(2) $(g_n\colon n \in \mathbb{N})$ converges pointwise to the zero function, and

(3) $(f_j^k\colon j \geqslant n_k)$ is a subsequence of $(g_n\colon n \in \mathbb{N})$.

Define a sequence $(\varepsilon_j\colon j \in \mathbb{N})$ so that for each j, if $n_k \leqslant j < n_{k+1}$, then $\varepsilon_j = (1/2)^k$; for $j < n_1$, put $\varepsilon_j = 1$.

Consider an $x \in X$. Fix N_0 so large that for each $n \geqslant N_0$, $|g_n(x)| < 1/2$. Then fix K so large that for each $k \geqslant K$ and for each $j \geqslant n_k$ there is an $m \geqslant N_0$ such that $f_j^k = g_m$. Thus, for all $k \geqslant K$ and for all $j \geqslant n_k$, $f_j^k(x) < 1/2$. This implies that for each $j \geqslant n_K$, $|f_j(x)| < \varepsilon_j$. We have shown that $(f_n\colon n \in \mathbb{N})$ converges to the zero function quasi-normally. □

A number of examples from the literature can now be used to compare the α_1-property with the α_2-property and the Fréchet property in the context of spaces of the form $\mathsf{C}_p(X)$ with X a subspace of the real line. For this we need the following concept: an open cover $\mathcal{U}$ of a space X is a γ-cover if it is infinite and each element of X is in all but finitely many elements of $\mathcal{U}$. It may be assumed that X itself is not a member of a given γ-cover. The symbol Γ denotes the collection of all γ-covers of X.

Corollary 5. *It is consistent, relative to the consistency of classical mathematics, that there is a set X of real numbers such that $\mathsf{C}_p(X)$ is an α_2-space but not an α_1-space.*

Proof. It was shown in [19] that if X is a set of real numbers which has property $\mathsf{S}_1(\Gamma,\Gamma)$, then $\mathsf{C}_p(X)$ has property α_2. In [10] it was shown that there exists an uncountable set of real numbers which has property $\mathsf{S}_1(\Gamma,\Gamma)$ (that it actually has this property was pointed out in [19]). In [16] it was shown that if X is a set of real numbers with the property QN, then X is a σ-set; this means that every F_σ-subset of X is also a G_δ-subset. In [12] Miller showed that it is consistent, relative to the consistency of classical mathematics, that no σ-set of real numbers is uncountable. □

Since it seems to be of particular interest to determine if one can outright prove whether there could be a set X of real numbers for which $\mathsf{C}_p(X)$ has property α_2 but not property α_1, it is useful to determine the exact axiomatic circumstances leading to the existence of the sorts of examples found in the literature. The next few results are motivated by these considerations. First, we rework the proof of Corollary 5 by extracting from the proof in [10] that there is an uncountable set of real numbers with property $\mathsf{S}_1(\Gamma,\Gamma)$, a little more information. A few more concepts are needed.

An open cover of a space is an ω-cover if the space itself is not a member of the cover, and each finite subset of the space is covered by some member of the cover. The symbol Ω denotes the set of ω-covers of a space. For f and g functions from $\mathbb{N}$ to $\mathbb{N}$, the symbol $f \prec g$ denotes that $\lim_{n\to\infty}(g(n) - f(n)) = \infty$. The binary relation $\prec$ is a partial ordering. The minimal cardinality of an unbounded subset for this order is denoted $\mathfrak{b}$. It is well known that $\mathfrak{b}$ is uncountable. For A and B infinite sets write $A \subset^* B$ to denote that $B\setminus A$ is infinite while $A\setminus B$ is finite. Let κ be an infinite cardinal number. A family $\{A_\alpha\colon \alpha < \kappa\}$ of infinite subsets of $\mathbb{N}$ is said to be a *tower* if it has the following properties: For $\alpha < \beta < \kappa$, $A_\beta \subset^* A_\alpha$, and there is no infinite set T such that for all $\alpha < \kappa$, $T \subset^* A_\alpha$. Towers exist. The minimal value of κ for which a tower exists is denoted $\mathfrak{t}$. It is well known that $\mathfrak{t}$ is uncountable.

Theorem 6. *If $\mathfrak{b} = \mathfrak{t}$, then there is an $\mathsf{S}_1(\Gamma,\Gamma)$-set of real numbers of cardinality $\mathfrak{b}$ such that no subset of it of cardinality $\mathfrak{b}$ is a QN-set.*

Proof. Let κ denote $\mathfrak{b}$ and $\mathfrak{t}$. Let $(f_\alpha\colon \alpha < \kappa)$ be a sequence in ${}^{\mathbb{N}}\mathbb{N}$ such that for $\alpha < \beta$ we have $f_\alpha \prec f_\beta$, and for each g in ${}^{\mathbb{N}}\mathbb{N}$ there is an α such that $\{n\colon g(n) < f_\alpha(n)\}$ is infinite. Recursively choose infinite subsets X_α, $\alpha < \kappa$ of $\mathbb{N}$ such that if $\alpha < \beta$, then $X_\beta \subset^* X_\alpha$, and for each α, the enumeration function $\mathrm{enum}(X_\alpha)$ of X_α eventually dominates f_α.

As in Claim 5.2 of [10] it follows that for each infinite subset S of $\mathbb{N}$ there is an $\alpha < \kappa$ such that the set

$$\big\{n\colon \big|S \cap \big[\mathrm{enum}(X_\alpha)(n), \mathrm{enum}(X_\alpha)(n+1)\big)\big| \geqslant 2\big\}$$

is infinite.

Let S be a subset of κ which is of cardinality κ. If we now set

$$X(S) := \{X_\alpha\colon \alpha \in S\} \cup [\mathbb{N}]^{<\aleph_0},$$

then as in Claim 5.3 of [10] one finds that for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of ω-covers of $[\mathbb{N}]^{<\aleph_0}$ there are: an infinite subset A of $\mathbb{N}$, an $\alpha \in S$, and a sequence $(V_n\colon n \in A)$ where for each $n \in A$ we have $V_n \in \mathcal{U}_n$, such that whenever $\beta \geqslant \alpha$ is in S, then for all but finitely many $n \in A$ we have $X_\beta \in V_n$.

It follows that the countable subset $[\mathbb{N}]^{<\aleph_0}$ of $X(S)$ is not a G_δ-subset of $X(S)$. Since by a result of [16] every F_σ (and thus every countable) subset of a QN-set is also a G_δ-set, $X(S)$ is not a QN-set. Put $X := X(\kappa)$. It further follows that if $(\mathcal{U}_n\colon n \in \mathbb{N})$ is a sequence of γ-covers of X then there are a sequence $(U_n\colon n \in \mathbb{N})$ and a subset Y of X with $|Y| < \kappa$ such that $U_n \in \mathcal{U}_n$ for each n, and $\{U_n\colon n \in \mathbb{N}\}$ is a γ-cover of $X \setminus Y$.

This implies that for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of γ-covers of X there is a sequence $(\mathcal{V}_n\colon n \in \mathbb{N})$ and a set $Y \subset X$ such that:

(1) $|Y| < \kappa$;
(2) For each n, $\mathcal{V}_n$ is an infinite subset of $\mathcal{U}_n$, and
(3) For each sequence $(V_n\colon n \in \mathbb{N})$ where for each n we have $V_n \in \mathcal{V}_n$, the set $\{V_n\colon n \in \mathbb{N}\}$ is a γ-cover of $X \setminus Y$.

To see this, write $\mathbb{N} = \bigcup_{n\in\mathbb{N}} Y_n$ where each Y_n is infinite, and any two of them are mutually disjoint. Then for each n choose an infinite $\mathcal{W}_n \subset \mathcal{U}_n$, such that any two $\mathcal{W}_n$'s are mutually disjoint. Then, for each n, write $\mathcal{W}_n = \bigcup_{k\in Y_n} \mathcal{S}_k$, where any two $\mathcal{S}_k$'s are disjoint, and each is infinite. Applying the preceding remark to the sequence $(\mathcal{S}_k\colon k \in \mathbb{N})$ of γ-covers of X, we find for each k an $S_k \in \mathcal{S}_k$, and we find a subset Y of X with $|Y| < \kappa$, such that $(S_k\colon k \in \mathbb{N})$ is a γ-cover of $X \setminus Y$. For each n define $\mathcal{V}_n := \{S_k\colon k \in Y_n\}$.

Finally, we see that the preceding remark implies that X has property $\mathsf{S}_1(\Gamma, \Gamma)$ as follows: Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of γ-covers of X. Choose a set $Y \subset X$ of cardinality less than κ, and for each n choose an infinite set $\mathcal{V}_n \subset \mathcal{U}_n$ as above. Since each $\mathcal{V}_n$ is a γ-cover of X, it is also a γ-cover of Y. Since the cardinality of Y is less than $\mathfrak{b}$, Theorem 4.7 of [10] implies that Y has property $\mathsf{S}_1(\Gamma, \Gamma)$. Thus, choose for each n, a $U_n \in \mathcal{V}_n$ such that $\{U_n\colon n \in \mathbb{N}\}$ is a γ-cover of Y. Then $\{U_n\colon n \in \mathbb{N}\}$ is a γ-cover of X. $\square$

Theorem 6 gives a slight strengthening of Theorem 5.1 of [10]:

Corollary 7. *There is a set of real numbers of cardinality* $\mathfrak{t}$ *which has property* $\mathsf{S}_1(\Gamma, \Gamma)$, *but is not* σ*-compact.*

Proof. It is well known that $\mathfrak{t} \leqslant \mathfrak{b}$. Now use Theorem 4.7 of [10], and Theorem 6. With a little more work one can show that the set X constructed in Theorem 6 also has property $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$. To see that the X obtained in Theorem 6 is not σ-compact, we need to concern ourselves only with the case when $\mathfrak{t} = 2^{\aleph_0}$. Notice that if Y is a Borel set of cardinality $2^{\aleph_0}$, and if $B \subset Y$ is countable, then $Y \setminus B$ contains an uncountable perfect set, and so there is an open set $U \supseteq B$ such that $Y \setminus U$ has cardinality $2^{\aleph_0}$. Since the countable subset $[\mathbb{N}]^{<\aleph_0}$ of X does not have this property relative to X, we see that X does not contain a perfect set of real numbers. $\square$

According to [9] a set of real numbers is a γ-set if it has property $\mathsf{S}_1(\Omega, \Gamma)$. The importance of this concept lies in the fact that a set X of real numbers is a γ-set if, and only if, $\mathsf{C}_p(X)$ has the Fréchet property. We shall now compare the α_1-property and the Fréchet property for $\mathsf{C}_p(X)$ when X is a set of real numbers.

In the proof of the next result we use another combinatorial concept: A collection of infinite subsets of $\mathbb{N}$ has the finite intersection property if each nonempty finite subcollection of it has nonempty intersection. An infinite set A is said to be a pseudo-intersection for a family $\mathcal{A}$ of infinite sets if for each $B \in \mathcal{A}$ we have $A \subset^* B$. A tower is an example of a family of infinite subsets of $\mathbb{N}$ which has no pseudo-intersection. The symbol $\mathfrak{p}$ denotes the least cardinal number κ for which there is a family of κ many infinite subsets of $\mathbb{N}$ which has the finite intersection property, but which does not have a pseudo-intersection. It is evident from the definitions that $\mathfrak{p} \leqslant \mathfrak{t}$; it is a notorious open problem whether one can in fact prove that $\mathfrak{p} = \mathfrak{t}$.

Corollary 8. *It is consistent that there is a set X of real numbers such that $\mathsf{C}_p(X)$ is Fréchet but not α_1.*

Proof. In Theorem 6.4 of [6] the authors show that if $\mathfrak{p} = 2^{\aleph_0}$, then there is a set X of real numbers which has property $\mathsf{S}_1(\Omega, \Gamma)$, but which is not a QN-set. Then by Theorem 2 of [9] $\mathsf{C}_p(X)$ is a Fréchet space. By Theorem 4 $\mathsf{C}_p(X)$ is not an α_1-space. $\square$

According to [3] a space X is an A_2-space if for every Borel function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$ there is a function g in ${}^{\mathbb{N}}\mathbb{N}$ such that for all $x \in X$, $\Psi(x) \prec g$.

Proposition 9. *If a set X of real numbers is an A_2-space, then $\mathsf{C}_p(X)$ is an α_1-space.*

Proof. Let X be a set of real numbers which also has property A_2. For each n let $(f_k^n\colon k \in \mathbb{N})$ be a sequence in $\mathsf{C}_p(X)$ which converges pointwise to the zero function.

For each n and each $x \in X$, define $\Psi_n(x)$ so that for each m

$$\Psi_n(x)(m) = \min\left\{k\colon\ \ell \geqslant k \Rightarrow |f_\ell^n(x)| < \frac{1}{m}\right\}.$$

Each Ψ_n is a Borel function from X to ${}^{\mathbb{N}}\mathbb{N}$. Since X is an A_2-space, there is for each n, a g_n such that for all x, $\Psi_n(x) \prec g_n$. Define g so that for each k

$$g(k) = \max\{g_i(j)\colon\ i, j \leqslant k\} + k.$$

For each n we have $g_n \prec g$. Thus g is such that for each x and for each n, $\Psi_n(x) \prec g$. Now define Φ from X to ${}^{\mathbb{N}}\mathbb{N}$ as follows: For each x and each n,

$$\Phi(x)(n) = \min\{k\colon\ j \geqslant k \Rightarrow \Psi_n(x)(j) < g(j)\}.$$

Then Φ is a Borel mapping, and so we may choose an h such that h is strictly increasing, $g \prec h$, and for each $x \in X$, $\Phi(x) \prec h$. For each n choose $k_n > 1$ so large that $h(n) < g^{k_n}(n)$, the k_nth iterate of g computed at n.

Then for each $\varepsilon > 0$, there exists for each $x \in X$ an $M \in \mathbb{N}$ such that

(1) for each $n \geqslant M$, for each $m \geqslant g^{k_n+1}(n)$, $|f_m^n(x)| < \varepsilon$, and

(2) for each $n < M$, for all but finitely many m, $|f_m^n(x)| < \varepsilon$.

Thus, the sequences $(f_j^n\colon j \geqslant g^{k_n+1}(n))$, $n \in \mathbb{N}$, witness the α_1-property of $\mathsf{C}_p(X)$. □

Corollary 10. *The minimal cardinality for a set X of real numbers such that $\mathsf{C}_p(X)$ does not have property α_1 is $\mathfrak{b}$.*

Proof. The minimal cardinality of a set of real numbers not having the A_2-property is $\mathfrak{b}$, and the minimal cardinality of a set of real numbers not having property $\mathsf{S}_1(\Gamma, \Gamma)$ is also $\mathfrak{b}$. □

A set X of real numbers is said to be a Sierpiński set if it has cardinality $2^{\aleph_0}$, and its intersection with any set of Lebesgue measure zero is uncountable. Sierpiński [20] proved that the Continuum Hypothesis implies the existence of a Sierpiński set.

Corollary 11. *If X is a Sierpiński set then $\mathsf{C}_p(X)$ has property α_1.*

Proof. It was shown in Theorem 2.9 of [10] that every Sierpiński set of real numbers is an A_2-space. □

Kunen [11] proved that for each infinite cardinal number κ it is consistent that $2^{\aleph_0} \geqslant \kappa$, and there is a Sierpiński set. Typically, models for this are obtained by starting with a model of the Continuum Hypothesis, and then adding a sufficient number of random reals side-by-side. In the final models obtained thus, one also has $\mathfrak{b} = \aleph_1$. Thus, it is entirely possible that there be sets of real numbers for which the corresponding function space is an α_1-space, and the cardinality of the set exceeds $\mathfrak{b}$.

Corollary 12. *It is consistent that there is a set X of real numbers for which $\mathsf{C}_p(X)$ is an α_1-space, but not a Fréchet space.*

Proof. (Proof 1) It is consistent that $\mathfrak{p} < \mathfrak{b}$. Then there is a set X of real numbers which does not have property $\mathsf{S}_1(\Omega, \Gamma)$, but is of cardinality less than $\mathfrak{b}$.

(Proof 2) Sierpiński sets do not have property $\mathsf{S}_1(\Omega, \Gamma)$. □

3. Comparison with other properties

If X is uncountable then $\mathsf{C}_p(X)$ is not first-countable, and thus sequences are not sufficient to describe the closure operator of $\mathsf{C}_p(X)$. Several weakened forms of the sequential description have been considered in this setting. When X is a set of real numbers, then $C_p(X)$ has countable tightness. This is an easy consequence of a theorem of Arhangel'skiĭ and (independently) Pytkeev—according to this theorem $C_p(X)$ has countable tightness if, and only if, all finite powers of X are Lindelöf.

We have seen that for X a set of real numbers one has:

(1) $C_p(X)$ has property α_2 if, and only if, it has property α_4;

(2) $C_p(X)$ could have property α_1 while not being Fréchet;

(3) $C_p(X)$ could be Fréchet while not having property α_1;

(4) if $C_p(X)$ has the Fréchet property, then it is α_2.

According to Sakai [17] a topological space has *countable strong fan tightness* if for each point x the selection hypothesis $S_1(\Omega_x, \Omega_x)$ is true. According to Gerlits and Nagy [9] topological space has the *strict Fréchet* property if for every point x the selection hypothesis $S_1(\Omega_x, \Gamma_x)$ holds. Closely related to this is the notion of a *strongly Fréchet* space: According to Siwiec [21] a space is strongly Fréchet if in the definition of strictly Fréchet we also require that the sequence of O_n's be monotonic. According to Arhangel'skiĭ [2] a space has countable fan tightness if for each point x the selection hypothesis $S_{\mathrm{fin}}(\Omega_x, \Omega_x)$ holds.

It is relatively easy to show that an α_1-space need not have countable tightness. For let X be an arbitrary α_1-space, and let Y be a space which is not countably tight, and has no convergent sequences (an uncountable set with the co-countable topology would do, but less pathological examples can be found). Then the topological sum $X+Y$ is an α_1-space which is not countably tight.

Gerlits and Nagy showed in [9] that for X a $T_{3\frac{1}{2}}$-space, $C_p(X)$ does not distinguish between the Fréchet properties: such a space has the Fréchet property if, and only if, it has the strict Fréchet property. A crucial part of this proof is the characterization of the Fréchet property of $C_p(X)$ in terms of the covering property $S_1(\Omega, \Gamma)$ of X.

The tightness properties of $C_p(X)$ have also been characterized in terms of covering properties of X: A result of Arhangel'skiĭ and Pytkeev does this for countable tightness, a result of Arhangel'skiĭ does this for countable fan tightness, and a result of Sakai does this for countable strong fan tightness. Due to these characterizations and results of [10] it has been shown that $C_p(X)$ distinguishes the tightness properties, even for X sets of real numbers.

As for the product theory of these classes: All these properties are preserved by finite powers of spaces of the form $C_p(X)$. The properties α_1 and α_2 are preserved by finite products. Due to examples of Przymusin'skiĭ and due to the Arhangel'skiĭ–Pytkeev theorem, there are spaces X and Y such that both $C_p(X)$ and $C_p(Y)$ have countable tightness, but $C_p(X) \times C_p(Y)$ does not have countable tightness. More recently Todorčević [22] even found examples of X and Y such that $C_p(X)$ and $C_p(Y)$ are Fréchet spaces, but $C_p(X) \times C_p(Y)$ does not have countable tightness. In all these cases the spaces X and Y are $T_{3\frac{1}{2}}$, but are not subspaces of the real line. Indeed, if X and Y are subspaces of the real line then $X+Y$ is still second countable, as is each finite power of it, so that by the Arhangel'skiĭ–Pytkeev theorem $C_p(X) \times C_p(Y)$ has countable tightness. But Todorčević also showed that it is consistent that there are subsets X and Y of the real line such that $C_p(X)$ and $C_p(Y)$ have the Fréchet property, while

$\mathsf{C}_p(X) \times \mathsf{C}_p(Y)$ does not have the Fréchet property (these examples are given after Theorem 5 of [8]).

The following diagram indicates the distinct classes of spaces that can be realized by $\mathsf{C}_p(X)$ for X a set of real numbers. The property listed at the origin of a vector implies the property at its endpoint.

$$\begin{tikzcd}
\alpha_1 \arrow[r] & \alpha_2 \arrow[rr] & & \text{Countably tight} \\
& \text{Fréchet} \arrow[u] \arrow[r] & \mathsf{S}_1(\Omega_x, \Omega_x) \arrow[r] & \mathsf{S}_{\text{fin}}(\Omega_x, \Omega_x) \arrow[u]
\end{tikzcd}$$

4. Problems

These results leave us now with a number of unresolved questions. The two most glaring ones seem to be as follows:

Problem 1. Could one prove in ZFC that there is a set X of real numbers for which $\mathsf{C}_p(X)$ has property α_2, but not property α_1?

Problem 2. Is it true that if a set X of real numbers has property QN, then the function space $\mathsf{C}_p(X)$ has property α_1?

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